3°. The theorem proved above with allowance for (32) implies the existence and uniqueness of second approximation solution of problem (12) – (15).

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ON THE EQUATIONS OF MOTION OF A LIQUID WITH BUBBLES

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An arbitrary irrotational flow of perfect incompressible liquid containing a considerable number of spherical gas bubbles is considered. Two methods of averaging exact characteristics of the motion of bubbles in the liquid, viz. by volume and by bubble centers, are introduced. Formulas relating the average quantities of two different kinds are derived. The boundary value problem for the mean potential is formulated on the basis of the exact boundary value problem for the velocity potential. The obtained equation for the potential in the particular case of unbounded liquid with low concentration of bubbles coincides with that derived in /1/.

It is shown that dynamic equations for the average characteristics of moving bubbles accurate to within the product of volume concentration by the velocity

of bubbles cannot be derived without considering the pattern of disposition of bubbles relative to the medium microstructure.

Closed equations of motion are derived for a liquid with bubbles, and conditions of the applicability of the model of liquid with "frozen in" bubbles are obtained.

A comprehensive survey of publications on the subject of equations of motion of liquid with bubbles appears in /1/.

1. The exact "macroscopic" problem. Two methods of averaging. An arbitrary surface S_0 and spheres S_α $(\alpha = 1, 2, \ldots)$ move in a potential stream of perfect incompressible fluid that is at rest at infinity. The fluid velocity field $\nabla\Phi$ is uniquely defined by the specified radii R_α, coordinates of the sphere centers $\mathbf{q}_\alpha$, and velocities $R_\alpha^\cdot$ and $\mathbf{q}_\alpha^\cdot$. The potential is determined by the solution of the Neumann problem

$$\Delta\Phi = 0; \quad \Phi \to 0, \quad r \to \infty \tag{1.1}$$

$$\left.\frac{\partial\Phi}{\partial n}\right|_{S_0} = v_n, \quad \left.\frac{\partial\Phi}{\partial n}\right|_{S_\alpha} = -R_\alpha^\cdot + \mathbf{q}_\alpha^\cdot \mathbf{n}_\alpha$$

where v_n is the normal translation velocity of surface S_0, whose direction is outward from the fluid.

Let the characteristic dimension l of surface S_0 considerably exceed $\max R_\alpha = R_+$. It is possible to determine dimension a that satisfies conditions $R_+ \ll a \ll l$, and carry out the averaging of basic parameters in a sphere of radius a.

If the density of gas, which is small in comparison with the density ρ of the liquid is neglected, the mean density ρ_* of the medium is

$$\rho_* = \int_{\Omega_f} \rho\Pi(\mathbf{x}-\mathbf{x}')\, d^3x', \quad \Pi(\mathbf{x}) = \begin{cases} 1/V, & |\mathbf{x}| \leqslant a \\ 0, & |\mathbf{x}| > a \end{cases}, \quad V = {}^4/_3\pi a^3 \tag{1.2}$$

where Ω_f is the total volume occupied by the liquid; it is bounded by surfaces S_0 and S_α. Density ρ_* is expressed in terms of volume concentration of bubbles c by

$$\rho_* = \rho(1-c), \quad c = \sum_\alpha \Pi(\mathbf{x}-\mathbf{q}_\alpha) V_\alpha, \quad V_\alpha = \frac{4}{3}\pi R_\alpha^3 \tag{1.3}$$

We define the averaging operations for functions $g(\mathbf{x})$ determinate in Ω_f and the discrete functions g_α specified the centers of bubbles as follows:

$$\langle g\rangle = \frac{1}{1-c}\int_{\Omega_f} \Pi(\mathbf{x}-\mathbf{x}') g(\mathbf{x}')\, d^3x', \quad \bar{g}_\alpha = \frac{1}{c}\sum_\alpha \Pi(\mathbf{x}-\mathbf{q}_\alpha) V_\alpha g_\alpha \tag{1.4}$$

It is assumed that the distribution of bubbles in space and the basic kinematic characteristics $\mathbf{q}_\alpha$, R_α, $R_\alpha^\cdot$ and $\mathbf{q}_\alpha^\cdot$ are such that there exists an averaging dimension a for which within distances of its order of magnitude the mean quantities (1.2) – (1.4) vary only little. This means that in any sphere of radius a within which averaging is carried out the number of bubbles is reasonably great.

The results of averaging is then, obviously, independent of the shape of the region in which it is carried out and of the particular form of the averaging function $\Pi(\mathbf{x})$. The sphere was chosen for simplicity of calculations.

Variation of the medium density (1.2) with time is completely determined by the normal velocity component of the liquid $\mathbf{v}'$ at its boundary

$$\frac{\partial \rho_*}{\partial t} = \int_{\partial\Omega_f} \rho \Pi (\mathbf{x} - \mathbf{x}') \mathbf{v}'\mathbf{n} dS = \int_{\Omega_f} \rho v_i' (\mathbf{x}') \frac{\partial}{\partial x_i'} \Pi (\mathbf{x} - \mathbf{x}') d^3x' \qquad (1.5)$$

Expressing the derivative of Π with respect to x_i' in terms of the derivative with respect to x_i, we obtain from (1.2) and (1.5) the equation of conservation of mass of the medium

$$\partial \rho_* / \partial t + \operatorname{div} \rho_* \mathbf{v} = 0, \quad \mathbf{v} = \langle \mathbf{v}' \rangle \qquad (1.6)$$

Differentiating both parts of the second formula in (1.3) with respect to time, we obtain the equation of continuity for the gas phase

$$\frac{\partial c}{\partial t} = 3c \frac{R^\cdot}{R} - \operatorname{div} c\mathbf{u}, \quad \frac{R^\cdot}{R} = \left(\overline{\frac{R_\alpha^\cdot}{R_\alpha}} \right), \quad \mathbf{u} = \overline{\mathbf{q}_\alpha^\cdot} \qquad (1.7)$$

It follows from (1.3), (1.6) and (1.7) that

$$\operatorname{div} [\mathbf{v} + c (\mathbf{u} - \mathbf{v})] = 3cR^\cdot / R \qquad (1.8)$$

Thus the derived method of averaging by formulas (1.3) and (1.4) yields, as expected, the known equations (1.6) and (1.7).

At the discontinuity surface moving at velocity D, the condition of conservation of mass is

$$[(1 - c) \ (\mathbf{vn} - D)] = 0 \qquad (1.9)$$

where brackets denote a jump of the function.

If the discontinuity moves together with the bubbles, i.e $D = \mathbf{un}$, then (1.9) assumes the form

$$[\mathbf{vn} + c (\mathbf{u} - \mathbf{v}) \mathbf{n}] = 0 \qquad (1.10)$$

Formula (1.10) is valid, for instance, at the surface separating the regions of fluid with and without bubbles.

2. Equations for average kinematic characteristics. Using the fundamental identity for harmonic functions, it is possible to represent the potential which is the solution of problem (1.1) in the form of a sum

$$\Phi = \sum_\alpha \Phi_\alpha, \quad 4\pi\Phi_\alpha = \int_{S_\alpha} \left(\frac{1}{r} \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n} \frac{1}{r} \right) dS, \qquad \alpha = 0, 1, \ldots \quad (2.1)$$

where Φ_0 is equal to the integral over surface S_0 of the moving body. Every function Φ_α is harmonic everywhere outside surface S_α. For $\alpha = 1, 2, \ldots$ function Φ_α is of the form

$$\Phi_\alpha = - \frac{R_\alpha^2 R_\alpha^\cdot}{|\mathbf{x} - \mathbf{q}_\alpha|} + \frac{1}{2} R_\alpha^3 w_{\alpha i} \frac{\partial}{\partial x_i} \frac{1}{|\mathbf{x} - \mathbf{q}_\alpha|} + \delta\Phi_\alpha \qquad (2.2)$$

Vector $\mathbf{w}_\alpha$ in (2.2), equal to the difference between the bubble velocity $\mathbf{q}_\alpha^\cdot$ and the stream velocity "adjusted" to the bubble center $\mathbf{q}_\alpha$, is defined by

$$\mathbf{w}_\alpha = \mathbf{q}_\alpha^\cdot - \nabla \Phi_\alpha' |_{\mathbf{x}=\mathbf{q}_\alpha}, \quad \Phi_\alpha' (\mathbf{x}) = \sum_{\beta \neq \alpha} \Phi_\beta (\mathbf{x}) \qquad (2.3)$$

The residual term $\delta\Phi_\alpha$ in (2.2) can be represented with the use of the Weiss theorem for a sphere /2/ as

$$\delta\Phi_\alpha (\mathbf{x}) = \frac{r_\alpha^*}{R_\alpha} \int_0^1 [\Psi^\cdot (\mathbf{r}_\alpha^*) - \Psi^\cdot (\mathbf{r}_\alpha^* t)] \, dt \qquad (2.4)$$

$$\Psi = \Phi_\alpha'(\mathbf{x}) - \Phi_\alpha'(\mathbf{q}_\alpha) - \frac{\partial \Phi_\alpha'}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{q}_\alpha} r_{\alpha i} \approx \frac{1}{2}\,\frac{\partial^2 \Phi'}{\partial x_i \partial x_j} r_{\alpha i} r_{\alpha j}$$

$$\mathbf{r}_\alpha = \mathbf{x} - \mathbf{q}_\alpha, \quad \mathbf{r}_\alpha{}^* = \frac{R_\alpha{}^2}{r_\alpha{}^2}(\mathbf{x} - \mathbf{q}_\alpha)$$

For the residual term in (2.2) from formula (2.4) we obtain

$$\delta\Phi_\alpha = \frac{R_\alpha{}^5}{9}\,\frac{\partial^2 \Phi'}{\partial x_i \partial x_j}\bigg|_{\mathbf{x}=\mathbf{q}_\alpha} \frac{\partial^2}{\partial x_i\, \partial x_j}\,\frac{1}{|\mathbf{x} - \mathbf{q}_\alpha|} + O\left(\frac{R_\alpha{}^5}{r_\alpha{}^4}\right)$$

Formulas (2.1) – (2.3) determine functions Φ and $\nabla\Phi$ which are continuous in Ω_f and, also, the discrete functions Φ_α' and $\nabla\Phi_\alpha'$ specified at the centers of spheres $\mathbf{q}_\alpha$. All these functions, which are kinematic characteristics of the medium, are uniquely determined by the solution of the boundary value problem (1.1), when the discrete functions R_α, $R_\alpha{}^\bullet$ and $\mathbf{q}_\alpha{}^\bullet$ and the normal velocity of boundary S_0 are known.

In conformity with the definitions (1.4) all these characteristics can be averaged. The averages $\langle\Phi\rangle$, $\langle\nabla\Phi\rangle$, $\overline{\Phi_\alpha'}$ and $\overline{\nabla\Phi_\alpha'}$ must be determined in terms of the averages of specified characteristics R_α, $R_\alpha{}^\bullet$, $\mathbf{q}_\alpha{}^\bullet$ and c. Below, the boundary value problem for $\langle\Phi\rangle$ is formulated and a method for computing the mean-mass velocity $\langle\nabla\Phi\rangle$ in terms of $\langle\Phi\rangle$ and of specified average characteristics is presented.

L e m m a 1. Formulas

$$\Phi(\mathbf{x}) = (1-c)\langle\Phi\rangle + c\overline{\Phi_\alpha'} + \sum_\alpha{}^\circ \Big(\Phi_\alpha(\mathbf{x}) - \frac{1}{V}\int_{V\setminus V_\alpha} \Phi_\alpha(\mathbf{x}')\, d^3x'\Big) \qquad (2.5)$$

$$\nabla\Phi(\mathbf{x}) = (1-c)\langle\nabla\Phi\rangle + c\overline{\nabla\Phi_\alpha'} + \sum_\alpha{}^\circ \Big(\nabla\Phi_\alpha(\mathbf{x}) - \frac{1}{V}\int_{V\setminus V_\alpha} \nabla\Phi_\alpha(\mathbf{x}')\, d^3x'\Big)$$

are valid to within smalls of the order of cR/a . In these formulas V is a sphere of radius a whose center is located at point $\mathbf{x}$, V_α is a sphere of radius R_α whose center lies at point $\mathbf{q}_\alpha$ (the region occupied by the bubble), $V\setminus V_\alpha$ is a region consisting of points of sphere V less the points belonging to sphere V_α. The symbol $\Sigma_\alpha{}^\circ$ represents the sum of only such terms to which $|\mathbf{q}_\alpha - \mathbf{x}| < a$. All functions in (2.5) are determined for some point $\mathbf{x} \in \Omega_f$.

Formulas (2.5) imply that the complete sum (2.1) is in the form of a "smoothed" function and of its deviation which is determined by exact values of potentials $\Phi_\alpha(\mathbf{x})$ related to spheres lying in the proximity of point $\mathbf{x}$. Note that the omission in sums (2.5) of terms for which boundaries of spheres V and V_α intersect each other is unimportant, since the contribution of their sums is small, being of order cR/a.

Both formulas (2.5) are derived in the same manner. The first of these can be obtained by using the definition (1.4) of the averaged value and formulas (2.1). We have

$$(1-c)\langle\Phi\rangle = \frac{1}{V}\sum_\alpha\Big(\int_{V\setminus V_\alpha} \Phi_\alpha{}'(\mathbf{x}')\, d^3x' - \sum_{\beta\neq\alpha}{}^\bullet \int_{V_\beta} \Phi_\alpha(\mathbf{x}')\, d^3x'\Big)$$

Taking into consideration that the integral of a harmonic function over a sphere is equal to the value of that function at the sphere center multiplied by the sphere volume and allowing for formulas (1.4), (2.1) and (2.3), we obtain

$$c\overline{\Phi_\alpha'} = \frac{1}{V}\sum_\alpha \sum_{\beta\neq\alpha}{}^\circ \int_{V_\beta} \Phi_\alpha(\mathbf{x}')\, d^3x'$$

Adding the last two formulas, after some simple transformations, we obtain the first of formulas (2.5). Lemma 1 is thus proved.

The integrals in (2.5) taken over region $V \setminus V_\alpha$ can be exactly computed by using expansion (2.2) and the following formulas:

$$\int\limits_{V\setminus V_\alpha} \frac{1}{|\mathbf{x}'-\mathbf{x}|} d^3x' = 2\pi\left(a^2 - R_\alpha^2 - \frac{1}{3}|\mathbf{q}_\alpha - \mathbf{x}|^2\right)$$
$$\int\limits_{V\setminus V_\alpha} \frac{\partial}{\partial x_i'} \frac{1}{|\mathbf{x}'-\mathbf{x}|} d^3x' = \frac{4\pi}{3}(q_{\alpha i} - x_i) \tag{2.6}$$

The related integrals of higher order multipoles vanish.

Lemma 1 makes it possible to establish the important relationship between the two kinds of parameters $\langle\Phi\rangle$ and $\overline{\Phi_\alpha'}$. To do this it is sufficient to average equality (2.5)

$$(1-c)(\overline{\Phi_\alpha'} - \langle\Phi\rangle) = \overline{Q_\alpha} \tag{2.7}$$
$$Q_\alpha(\mathbf{q}_\alpha) = \sum_{\beta\neq\alpha}^{\circ}\left(\Phi_\beta(\mathbf{q}_\alpha) - \frac{1}{V}\int\limits_{V\setminus V_\beta} \Phi_\beta(\mathbf{x}')\, d^3x'\right)$$

where V is a sphere of radius a whose center is at point $\mathbf{q}_\alpha$.

A similar formula is valid also for the potential gradient. With the use of (2.6) it can be presented in the form

$$(1-c)\left(\overline{\frac{\partial\Phi_\alpha'}{\partial x_i}} - v_i\right) = \overline{P_{\alpha i}}, \quad v_i = \left\langle\frac{\partial\Phi}{\partial x_i}\right\rangle \tag{2.8}$$
$$P_{\alpha i} = \frac{3}{8\pi}\sum_{\beta\neq\alpha}^{\circ} V_\beta \frac{\partial^2}{\partial q_{\beta i}\,\partial q_{\beta j}} \frac{1}{|q_\alpha - q_\beta|} w_{\beta j}$$

The estimates in (2.7) and (2.8) with allowance for (2.2) and (2.6) lead to the conclusion that

$$\overline{\Phi_\alpha'} - \langle\Phi\rangle = O\left(c\frac{R^{\cdot}}{R}a^2\right), \quad \overline{\nabla\Phi'} - \langle\nabla\Phi\rangle = O(c\overline{\mathbf{w}}_\alpha) \tag{2.9}$$

Since parameter a of averaging is "infinitely" small, the averages $\overline{\Phi_\alpha'}$ and $\langle\Phi\rangle$ are identical. If these functions are considered as generalized, then their derivatives of any order are also equal. The average velocities $\overline{\nabla\Phi_\alpha'}$ and $\langle\nabla\Phi\rangle$ are, however, generally substantially different.

If it is assumed, as in /1/, that $\mathbf{w}_\beta$ varies only little at distances of order a, then vector $\mathbf{w} \approx \mathbf{w}_\beta$ can be removed from the summation sign in (2.8)

$$(1-c)\left(\overline{\frac{\partial\Phi_\alpha'}{\partial x_i}} - v_i\right) = c\overline{A}_{ij} w_j \tag{2.10}$$
$$cA_{ij} = \frac{3}{8\pi}\sum_{\beta\neq\alpha}^{\circ} V_\beta \frac{\partial^2}{\partial q_{\beta i}\partial q_{\beta j}} \frac{1}{|\mathbf{q}_\alpha - \mathbf{q}_\beta|}$$

It will be readily seen that the trace of tensor A_{ij} is zero

$$A_{ii} = 0 \tag{2.11}$$

Terms which correspond to bubbles situated in the vicinity of $\mathbf{q}_\alpha$ provide the main contribution to the sum appearing in (2.10). If the variation of V_β inside the sphere is small, the sum of terms corresponding to bubbles situated in a spherical layer $r < |\mathbf{q}_\alpha - \mathbf{x}| < a$ at a reasonable distance from point $\mathbf{q}_\alpha$ can be approximated by an integral which is equal zero.

The average value of A_{ij} depends on the pattern of the relative disposition of bubbles (the microstructure). In the case of isotropy tensor $\bar{A}_{ij}$ is spherical, and in virtue of (2. 11) we have $\bar{A}_{ij} = 0$. It follows from this and (2. 10) that in the case of isotropy $\overline{\nabla\Phi_\alpha'} = \mathbf{v}$.

Let $f(\mathbf{x}, \mathbf{x}', t)$ be the probability density of finding a bubble at point $\mathbf{x}'$, if point $\mathbf{x}$ is already occupied by the center of a given bubble. Function f defines the relative disposition of adjacent bubbles, and is of the form

$$f(\mathbf{x}, \mathbf{x}', t) = F(\mathbf{x}, \mathbf{x} - \mathbf{x}', t)\, \alpha(\mathbf{x}') \tag{2. 12}$$

where $\alpha(\mathbf{x}')$ is the number of bubbles in a unit of volume. For $r = |\mathbf{x} - \mathbf{x}'| \gg r_0$, where r_0 is the average distance between bubbles, function F is equal unity (weakening of correlation), is regular with respect to the first argument, and varies at microscopic distance in accordance with the second argument. Tensor $\bar{A}_{ij}$ in (2. 10) in terms of probability density (2. 12) is expressed by

$$\bar{A}_{ij} = \frac{3}{8\pi} \int\limits_{|x-x'|<a} F(\mathbf{x}, \mathbf{x} - \mathbf{x}', t) \frac{\partial^2}{\partial x_i' \partial x_j'} \frac{1}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' \tag{2. 13}$$

If the lines of level $F = \text{const}$ represent the surfaces of similar concentric ellipsoids, then tensor $\bar{A}_{ij}$ is generally independent of the specific form of the correlation function F. It can be shown that in that case

$$\bar{A}_{ij} = \frac{3}{8\pi} \int\limits_G \frac{\partial^2}{\partial x_i' \partial x_j'} \frac{1}{r}\, d^3x' = \frac{3}{8\pi} \int\limits_S n_i \frac{\partial}{\partial x_j'} \frac{1}{r}\, dS - \frac{1}{2}\delta_{ij}$$

where G is the volume of space bounded from inside by surface S of an ellipsoid and from outside by the sphere of radius a; the integrals are independent of the selection of a particular ellipsoid from the set of similar ellipsoids.

The components of tensor $\bar{A}_{ij}$ are functions of five parameters: the ratios of the principal axes of the ellipsoid and of three Euler's angles which determine the position of the ellipsoid principal axes. Depending on these parameters, components of tensor $\bar{A}_{ij}$ can be arbitrarily great.

In the case of steady motion such as, for instance, bubbling, it is reaonable to assume that A_{ij} is a tensor function of the relative velocity $\mathbf{w}$. The general form of such tensor function satisfying condition (2. 11) is

$$A_{ij} = (3w_i w_j - w^2 \delta_{ij}) f(|\mathbf{w}|)$$

L e m m a 2. Formula

$$\nabla((1-c)\langle\Phi\rangle + \overline{c\Phi_\alpha'}) = (1-c)\langle\nabla\Phi\rangle + c\overline{\nabla\Phi_\alpha'} - \frac{1}{2}c\overline{\mathbf{w}}_\alpha \tag{2. 14}$$

is valid to within smalls of order cR/a.

P r o o f. By differentiating the integrals with respect to $\mathbf{x}$ of which the region of integration and transformation of integrands is independent, we obtain

$$\frac{\partial}{\partial x_i} \int\limits_{\Omega_f} \Pi(x - x')\, \Phi(\mathbf{x}')\, d^3x' = \int\limits_{\Omega_f} \left[\Pi \frac{\partial\Phi}{\partial x_i'} - \frac{\partial}{\partial x_i'}(\Pi\Phi)\right] d^3x'$$

$$\frac{\partial}{\partial x_i} \sum_\alpha \int\limits_{V_\alpha} \Pi(x - x')\, \Phi_\alpha'(\mathbf{x}')\, d^3x' = \sum_\alpha \int\limits_{V_\alpha} \left[\Pi \frac{\partial\Phi_\alpha'}{\partial x_i'} - \frac{\partial}{\partial x_i'}(\Pi\Phi_\alpha')\right] d^3x'$$

Using definition (1. 4) for averages and reducing the last integrals to surface integrals, we can obtain

$$\frac{\partial}{\partial x_i}(1-c)\langle\Phi\rangle = (1-c)\left\langle\frac{\partial\Phi}{\partial x_i}\right\rangle - \sum_\alpha \Pi(\mathbf{x}-\mathbf{q}_\alpha)\int_{S_\alpha}\Phi(\mathbf{x}')\,n_i\,dS \qquad (2.15)$$

$$\frac{\partial}{\partial x_i}(c\overline{\Phi_\alpha'}) = c\,\overline{\frac{\partial}{\partial x_i}\Phi_\alpha'} + \sum_\alpha \Pi(\mathbf{x}-\mathbf{q}_\alpha)\int_{S_\alpha}\Phi_\alpha'(\mathbf{x}')\,n_i\,dS$$

where the equalities are accurate to within terms corresponding to spheres V_α, whose boundaries intersect the surface of the sphere of radius a whose center is at point $\mathbf{x}$. The contribution of these terms is small, of order cR/a.

Formula (2. 14) is obtained by adding equalities (2. 15) and using the relationship

$$\Phi - \Phi_\alpha' = \Phi_\alpha, \quad \int_{S_\alpha}\Phi_\alpha \mathbf{n}\,dS = \frac{1}{2}\mathbf{w}_\alpha$$

From Lemma 2 with allowance for estimates (2. 9) follows the important corollary

$$\mathbf{v} = \nabla\langle\Phi\rangle + \frac{1}{2}c\mathbf{w} + O(c^2 w), \quad \mathbf{w} = \mathbf{u} - \nabla\langle\Phi\rangle, \quad \mathbf{v} = \langle\nabla\Phi\rangle \qquad (2.16)$$

where v is taken as the medium mean-mass velocity which appears in the equation of conservation of mass (1. 8).

Substituting (2. 16) into the equation of continuity (1. 8), for the averaged potential we obtain

$$\nabla\{(1-{}^3/_2\,c)\nabla\langle\Phi\rangle\} = 3cR^{\cdot}/R - {}^3/_2\,\nabla(c\mathbf{u}) + O(c^2) \qquad (2.17)$$

Condition

$$\mathbf{vn} = v_n \quad \text{or} \quad \frac{\partial}{\partial n}\langle\Phi\rangle = v_n + \frac{1}{2}c(v_n - \mathbf{un}) \qquad (2.18)$$

must be satisfied at the surface S_0 of the solid body.

We have thus proved the following theorem.

Theorem. Let Φ be a solution of the boundary value problem (1. 1), $\langle\Phi\rangle$ and $\mathbf{v} = \langle\nabla\Phi\rangle$ be functions obtained by averaging potential Φ and its gradient in conformity with (1. 4), c a function determined by (1. 3), and $R^{\cdot}/R$ and u functions determined by (1. 7). Then function $\langle\Phi\rangle$ satisfies Eq. (2. 17), the boundary condition (2. 18) and condition (1. 9) at the discontinuity, and v is determined by formula (2. 16).

Thus, if concentration c, the average velocity of bubble motion $\mathbf{u}$, and the average characteristic of variation of their volume $R^{\cdot}/R$ are known, the problem of computing the mean-mass velocity reduces to the solution of the boundary value problem (2. 17), (2. 18). Velocity $\mathbf{v}$ is then determined by formula (2. 16).

If the assumption made in the derivation of formula (2. 10) is true, it is possible to obtain from Lemma 2 the more accurate relationship than (2. 16)

$$v_i = \nabla_i\langle\Phi\rangle + {}^1/_2\,cw_i - c^2\bar{A}_{ij}w_j/(1-c) \qquad (2.19)$$

which is valid for any bubble volume concentration c.

Furthermore, if the relative disposition of bubbles is, in the average, isotropic, then, as shown previously, $\bar{A}_{ij} = 0$, and formula (2. 19) and the equation for $\langle\Phi\rangle$ are considerably simplified.

Formula (2. 19) is also valid in the particular case in which the bubble centers are at the nodes of a quasi-periodic lattice. For such structures tensor $\bar{A}_{ij}$ can be defined by parameters of the lattice.

Formulas (2.16) and (2.17) were previously obtained in /1/ by a different method. Unlike in this work, it was in addition assumed there that R_α, $R_\alpha^{\cdot}$ and $\mathbf{w}_\alpha$ are regular functions of coordinates (i.e. their variations are macroscopic), and that solid boundaries are absent. The assumption in /1/ that for almost all bubbles

$$\left.\frac{\partial \Phi_\alpha'}{\partial x_i}\right|_{x=\mathbf{q}_\alpha} = \frac{\partial}{\partial x_i}\langle\Phi\rangle \qquad (2.20)$$

is approximately true, the error being $\sim cw$. In fact, it follows from (2.10) and (2.19) that

$$\overline{\nabla_i \Phi_\alpha'} = \nabla_i \langle\Phi\rangle + {}^1/_2 c w_i + c\overline{A}_{ij} w_j \qquad (2.21)$$

and assumption (2.20) proves to be correct in the trivial case of $\mathbf{w} = 0$ or, if simultaneously the determinant of matrix $\delta_{ij} + 2\overline{A}_{ij}$ is zero and vector $\mathbf{w}$ is the eigenvalue of matrix $\overline{A}_{ij}$. For an isotropic structure or one close to it (2.20) is satisfied only for $\mathbf{w} = 0$.

Example 1. Let us consider the problem of the velocity field for the motion of a bubble cloud in an unbounded liquid. Let concentration c of bubbles in the volume of a sphere of radius l be constant, while outside it is zero. Let the velocities of all bubbles be the same and equal $\mathbf{u}$. Then Eq. (2.17) assumes the form $\Delta\langle\Phi\rangle = 0$.

At the sphere boundary the average potential is continuous and the normal derivative in conformity with (1.10) becomes discontinuous. Within smalls of order c the discontinuity is defined by

$$[\partial\langle\Phi\rangle / \partial n] = -{}^3/_2\,[c\mathbf{un}] = {}^3/_2\, c\mathbf{un}$$

The solution of the problem is of the form

$$\langle\Phi\rangle = -{}^1/_2\, cur\cos\theta,\ \ v = 0 \qquad \text{for } r < l$$

$$\langle\Phi\rangle = -{}^1/_2\, cu\,(l^3/r^2)\cos\theta,\ \ \mathbf{v} = \nabla\langle\Phi\rangle \quad \text{for } r \geqslant l$$

where r and θ are spherical coordinates with origin at the sphere center; angle θ is measured from the line of the velocity vector $\mathbf{u}$. The potential $\langle\Phi\rangle$ in the case of motion of a spherical cloud of bubbles differs from that of motion of a solid sphere of radius l by the factor c.

The motion of an ellipsoidal cloud of bubbles was considered in /3, 4/ on the assumption that the bubble centers were located at nodes of a cubic lattice. The average velocity $\overline{\nabla\Phi_\alpha'}$ of the liquid induced at the bubble center and the rate of rise of the cloud in a heavy liquid of low viscosity were computed by the Lorentz method. The same results can be obtained by the method expounded here, if one takes into consideration that the cubic lattice structure is isotropic and $\overline{\nabla\Phi_\alpha'} = \langle\nabla\Phi\rangle$. The problem reduces to solving the internal and external Neumann problem for an ellipsoid.

Example 2. Let bubbles rise from a small area S of a horizontal area at constant velocity u, filling a cylindrical column with S as its base. The boundary value problem for the average potential is of the form: $\Delta\langle\Phi\rangle = 0$; $\partial\langle\Phi\rangle/\partial x = -{}^1/_2\, cu$ over S; $\langle\Phi\rangle \to 0$ and $r \to \infty$. In that case the derivatives of the potential are continuous at the column boundary. There is an analogy between velocity field of the liquid outside the bubbles and that of the flow of a perfect incompressible liquid from an orifice of area S at the rate of ${}^1/_2\, cu$.

3. The equations of motion of a liquid with bubbles. To compute the quantities averaged above over the volume it is necessary to examine the bubble velocities $\mathbf{q}_\alpha^{\cdot}$ and the rates of change of radii $R_\alpha^{\cdot}$.

Let r_0 (the distance of the considered bubble to the one next to it) be considerably greater than the bubble radius R. The following equations are valid to within smalls of an order not lower than $(R / r_0)^6$ for the bubble motion in a nonuniform stream /5 —7/, induced by the motion of other bubbles and solid bodies:

$$q_{\alpha i}^{\cdot\cdot} = 3\left(\frac{\partial}{\partial t} + \frac{\partial \Phi_\alpha'}{\partial x_j}\frac{\partial}{\partial x_j}\right)\frac{\partial \Phi_\alpha'}{\partial x_i} - 2\frac{\partial U}{\partial x_i} - \frac{3R_\alpha^{\cdot}}{R_\alpha}\left(q_{\alpha i}^{\cdot} - \frac{\partial \Phi_\alpha'}{\partial x_i}\right) \tag{3. 1}$$

$$R_\alpha^{\cdot\cdot} R_\alpha + \frac{3}{2} R_\alpha^{\cdot 2} - \frac{1}{4}|\mathbf{q}_\alpha^{\cdot} - \nabla\Phi_\alpha'|^2 = \frac{p_g - p_\alpha'}{\rho} - \frac{2\sigma}{\rho R_\alpha}$$

$$p_\alpha' + \rho(\partial\Phi_\alpha'/\partial t + {}^1/_2(\nabla\Phi_\alpha')^2 + U) = \mathrm{const}$$

where U is the potential of mass forces. This system yields $4N$ equations of motion for N bubbles in the system. The computation of Φ_α' for every configuration of bubbles represents a kinematic problem.

It is important to estimate the accuracy with which it is possible to average Eqs. (3. 1) of motion of bubbles. Let us assume that motion parameters vary only little over distances of order r_0. Then the values of any quantities in (3. 1) averaged with respect to bubble centers can be expressed in terms of probability density (2. 12). For instance, taking into account (2. 2) and (2. 3) it is possible to express the average potential Φ_α' and velocity $\overline{\nabla\Phi_\alpha'}$ averaged with respect to centers in the form

$$\overline{\Phi_\alpha'} = \Phi_0 + \int\limits_\Omega f(\mathbf{x}, \mathbf{x}', t)\left(-\frac{R^2R^{\cdot}}{r} + \frac{R^3 w_i}{2}\frac{\partial}{\partial x_i}\frac{1}{r}\right)d^3x' \tag{3. 2}$$

$$\overline{\nabla\Phi_\alpha'} = \nabla\Phi_0 + \int\limits_\Omega f(\mathbf{x}, \mathbf{x}', t)\left(-R^2R^{\cdot}\nabla\frac{1}{r} + \frac{R^3 w_i}{2}\frac{\partial}{\partial x_i}\nabla\frac{1}{r}\right)d^3x'$$

$$\mathbf{w} = \mathbf{u} - \overline{\nabla\Phi_\alpha'}$$

where $\mathbf{w}$ is the relative velocity of bubbles.

The integral formulas (3. 2) make it possible to establish the relation between the average gradient of the potential and the gradient of the average potential

$$\overline{\nabla_i\Phi_\alpha'} = \nabla_i(\overline{\Phi_\alpha'}) + {}^1/_2\, c w_i + c\bar{A}_{ij} w_j \tag{3. 3}$$

which in virtue of estimate (2. 9) is the same as the derived above relationship (2. 21). Tensor $\bar{A}_{ij}$ is determined by formula (2. 13).

The absolute convergence for $r \to 0$ of integral terms in (3. 2), the insignificance of function f at small distances $r \leqslant a$ for which function $\alpha(\mathbf{x}')$ can be substituted everywhere, can be used for the derivation of formula (3. 3).

The derivative with respect to x can be brought out from the integrand

$$\nabla\Phi_0 + \int\limits_\Omega f\left(-R^2R^{\cdot}\nabla\frac{1}{r}\right)d^3x' = \nabla\left(\Phi_0 + \int\limits_\Omega f\left(-R^2R^{\cdot}\frac{1}{r}\right)d^3x'\right) \tag{3. 4}$$

On the other hand, the integral in (3. 2) of the term of form $\nabla_i\nabla_j(1/r)$ is generally divergent for $r \to 0$, and its value substantially depends on the form of function f when $r \to 0$.

With the use of (2. 12) this integral can be represented in the form

$$\int_{\Omega} f \frac{R^3 w_j}{2} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} d^3x' = \frac{\partial}{\partial x_j} \int_{\Omega} f \frac{R^3 w_j}{2} \frac{\partial}{\partial x_i} \frac{1}{r} d^3x' - \int_{\Omega} \frac{\alpha R^3 w_j}{2} \frac{\partial F}{\partial x_j} \frac{\partial}{\partial x_i} \frac{1}{r} d^3x' \tag{3.5}$$

Further it is possible to use the identity

$$\frac{\partial}{\partial x_j} \int_{r \leqslant a} F \frac{\partial}{\partial x_i} \frac{1}{r} d^3x' = \int_{r \leqslant a} \frac{\partial F}{\partial x_j} \frac{\partial}{\partial x_i} \frac{1}{r} d^3x' + \int_{r \leqslant a} F \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} d^3x' + \int_{r = a} F \frac{\partial}{\partial x_i} \frac{1}{r} n_j dS$$

where the integral in the left-hand part is a small quantity of order a. The first integral in the right-hand part differs from the last integral in (3. 5) by the factor $\frac{1}{2} \alpha R^3 w_j$, since derivatives of F vanish outside the sphere of radius a. The second integral is, in accordance with formula (2. 13), equal to tensor $\bar{A}_{ij}$ to within the multiplicant. Finally, the integral over the sphere $r = a$ is readily determined, since over the sphere $F = 1$. As the result we obtain

$$-\int_{\Omega} \frac{\alpha R^3 w_j}{2} \frac{\partial F}{\partial x_j} \frac{\partial}{\partial x_i} \frac{1}{r} d^3x' = \left(\bar{A}_{ij} + \frac{1}{2} \delta_{ij} \right) c w_j$$

Substituting the derived relationship into (3. 5) and using formulas (3. 2) and (3. 4), we obtain (3. 3).

Tensor $\bar{A}_{ij}$, appearing in (3. 3) is determined by the form of the correlation function F that defines the average relative disposition of nearest bubbles. For instance, if the surface of level $F = \text{const}$ represents concentric spheres (isotropy), then it follows from (2. 13) that $\bar{A}_{ij} = 0$. When the surfaces of level are similar concentric ellipsoids, then tensor $\bar{A}_{ij}$ is nonzero and, owing to the property (2. 11), does not reduce to a spherical one.

As shown in Section 2, components of tensor $\bar{A}_{ij}$, depending on parameters of ellipsoids, can assume any values, and the last term with $\bar{A}_{ij} w_j$ in (3, 3) can be a quantity of the same order as $\frac{1}{2} c w_i$. Thus the average velocity $\overline{\nabla \Phi_\alpha'}$ of the external nonuniform stream depends on the average relative position of adjacent bubbles, and its terms of order $c\mathbf{w}$ can assume any values, depending on the microscopic structure of the system of bubbles in the liquid. These values do necessarily vary with time. For instance, if the velocities of bubble centers are values of the regular function $\mathbf{u}$ at related points, the surface of level $F = \text{const}$ varies with time in conformity with the affine transformation of a small element of the sphere.

The rate of strain of a small element of the medium is determined by the rate of strain tensor $\frac{1}{2} (\nabla_j u_i + \nabla_i u_j)$. It is possible to establish the relation between the variation of tensor $\bar{A}_{ij}$ and the strain rate tensor. Hence in this case the isotropic distribution of bubbles (the surfaces of level are spheres) are transformed during subsequent instants of time in conformity with strains of medium elements into an essentially anisotropic distribution (the surfaces of level become ellipsoids). This always occurs when $\mathbf{grad}\ \mathbf{u} \neq 0$. Hence it is not possible to allow in formula (3. 3) for terms of order $c\mathbf{w}$ without considering the evolution of the system.

It is possible to conclude that in the averaging of equations of motion (3. 1) all terms of order $c\mathbf{w}$ must be disregarded, since they cannot be taken into account without considering the problem about the average relative disposition of adjacent bubbles and its evolution with time. To take into account in equations of motion terms of the form $c\mathbf{w}$, it is necessary to consider some additional parameters that define the medium microstructure.

Let us assume that the external stream $\mathbf{v}'$ is essentially generated by the change of bubble radii and the motion of the solid body boundary S, while the contribution due to translational motion of bubbles is comparatively small. It is then possible to neglect in the integrals in (3. 2) the terms which depend on $\mathbf{w}$. Estimates show that such approximation is valid for $l\,|R^{\cdot}| \gg Rcw$, where l is the scale of the region in which variation of bubble radii takes place.

When small quantities of order cw are neglected, the average equations (3. 1) assume the form

$$u_i^{\cdot} = 3\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) - 2\frac{\partial U}{\partial x_i} - 3\frac{R^{\cdot}}{R}(u_i - v_i) \tag{3. 6}$$

$$RR^{\cdot\cdot} + \frac{3}{2}R^{\cdot 2} - \frac{1}{4}(\mathbf{u}-\mathbf{v})^2 = \frac{p_g - p}{\rho} - \frac{2\sigma}{\rho R}$$

$$p + \rho\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + U\right) = \text{const}, \quad \mathbf{v} = \nabla\varphi$$

where the dot denotes differentiation with respect to time along the bubble trajectory.

Potential φ of external velocity $\mathbf{v}$ is determined by the solution of the boundary value problem.

$$\Delta\varphi = 3\frac{R^{\cdot}}{R}c, \quad \left.\frac{\partial\varphi}{\partial n}\right|_S = v_n \tag{3. 7}$$

where S is the surface of the solid body and v_n is the normal velocity of the surface.

Problem (3. 7) differs from that for Eq. (2. 17) with boundary conditions (2. 18) by the absence of terms of order $c\mathbf{w}$. As shown above, the allowance for the latter when solving the problem of bubble motion is incorrect. Note that attempts at taking into account terms of the form $c\mathbf{w}$ are made in all latest investigations on equations for a liquid with bubbles (see the survey in /1/). However, such allowance is inadmissible without an investigation of the microstructure of a medium with bubbles. Terms of the form $c\mathbf{w}$ can be taken into consideration in formulas for the fluid velocity averaged over volume after solving the problem of bubble motion on the basis of Eqs. (3. 6) and (3. 7).

To close the system of equations it is sufficient to add to (3. 6) and (3. 7) Eq. (1. 7) for the variation of bubble concentration and the dependence of gas pressure p_g in the bubble on the pattern of its motion, in the simplest case $p_g = p_g(R)$.

The conclusion about the impossibility of taking into account the dynamic terms in equations with an accuracy to within the product $c\mathbf{w}$ of particle concentration by the relative velocity without investigating the average characteristics of the relative position of adjacent bubbles is, evidently, valid for any arbitrary multiphase medium, and not only for a liquid with bubbles.

Let us take into account in the equation of motion (3. 6) of a bubble the force of viscous drag

$$F_i = \frac{4}{3}\pi\rho R^3\frac{1}{\tau_0}w_i, \quad \tau_0 = \frac{R^2}{k\nu}$$

$$u_i^{\cdot} = 3\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) - 2\frac{\partial U}{\partial x_i} - \left(3\frac{R^{\cdot}}{R} + \frac{2}{\tau_0}\right)(u_i - v_i) \tag{3. 8}$$

where the dimensionless coefficient k can assume various values. For small Reynolds numbers $\mathrm{Re} = Rw / \nu \ll 1$ coefficient k can vary from $k = 3$, according to the Adamar-Rybchinskii solution for the motion of a bubble in pure liquid, to $k = {}^9/_2$ according to the Stokes solution, if owing to the presence of surface-active matter the bubble surface "hardens". At high Reynolds numbers $\mathrm{Re} \gg 1$, $k = 9$ in the case of bubble motion in a pure liquid.

The introduction of viscosity in Eq. (3.8) implies the assumption of the additivity of dynamic and viscous forces acting on a bubble. This hypothesis, widely used in the hydrodynamics of multiphase media, is in the general case incorrect. It can be, however, justified in the limit case of $\mathrm{Re} \to 0$ with the use of the Navier-Stokes equations and, also, for $\mathrm{Re} \gg 1$ with the use of boundary layer equations for the free surface.

Let the characteristic time of variation of the liquid particle velocity $\mathbf{v}$ be τ which does not exceed the characteristic time of radius variation. Then the estimates of terms of Eq. (3.8) make it possible to determine the characteristic value of the relative velocity $\mathbf{u} - \mathbf{v}$

$$|\mathbf{u}-\mathbf{v}| \ll |\mathbf{v}| \quad \text{for} \quad \tau \gg R^2/(k\nu) \qquad (3.9)$$
$$|\mathbf{u}-\mathbf{v}| \sim |\mathbf{v}| \quad \text{for} \quad \tau \lesssim R^2/(k\nu)$$

The first of conditions (3.9) is that of the "freezing in" of bubbles in the medium. Thus when a stream flows at velocity v past a body with the characteristic dimension l , the bubbles for which

$$R \ll \sqrt{k\nu l / v}$$

are those that are frozen in the medium. This estimate is also valid for a tube of cross section of scale l.

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